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Marking Scheme -Mathematical Olympiad – Stage I – 2016

Note: All alternate solutions are to be accepted at par. Please asses what a student knows in place of what he doesn't know.

1. **There are four numbers a,b,c and d such that a<b<c<d can be paired in six different ways. If each pair has a different sum, and if the four smallest sums are 1,2,3 and 4 , find all possible values of 'd'.**

Expected Soln.

Method I : The six possible sums are a+b,a+c,b+c,a+d,b+dand c+d . Since a<b<c<d, the smallest sums are a+b=1 and a+c= 2 from these c=b+1 (1+1)

Of the other four sums , the largest are b+d and c+d. Of the remaining sums, b+c and a+d, one equal to 3 and other equal to 4. (2)

So b+c=3 => b+b+1= 2b+1=3 gives b= 1 and since a+b=1, gives a=0 (1)

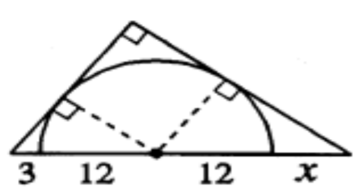
a+d= 4, gives d=4 as a=0,if instead b+c=2b+1=3 gives b=1 (1)

Then since a+b=1, a=0 and since a+d=4 and d=4 (1)

If instead b+c=2b+1=4 => b=3/2 and a+b=1, gives a=-1/2 since a+d=3 gives d= 7/2 (2)

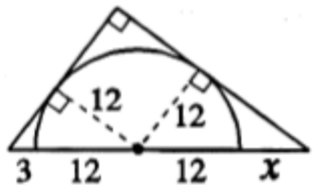
So two possible values of 'd' are 7/2 , 4 (1)

2. **A semicircle is tangent to both legs of a right triangle and has its centre on the hypotenuse. The hypotenuse is partitioned into 4 segments, with lengths 3, 12, 12,and *x*, as shown in the figure . Determine the value of '*x*' ?**



Soln.

The two small right $\triangle$s are similar to each other (and the large right $\triangle$). A radius of the circle is 12. Thus the longer leg



of the right $\triangle$ at the lower left is 12. Since its hypotenuse is 15, its dimensions are 9, 12, 15. The shorter leg of the right $\triangle$ at the lower right is 12, so its dimensions are 12, 16, 20. Since $12+x = 20$, $x = \boxed{8}$.

(To Show Dimension 9,12,15 &12,16,20—4marks each and x=8 (2marks))

3.(a) Find all three digit numbers *abc*(with $a \neq 0$) such that $a^2 + b^2 + c^2$, isdivisible by 26.

Soln.

Possible factors are 1, 2, 13, 26. Ignoring order, the possible expressions as a sum of three squares are: $1 = 1^2 + 0^2 + 0^2$, $2 = 1^2 + 1^2 + 0^2$, $13 = 3^2 + 2^2 + 0^2$, $26 = 5^2 + 1^2 + 0^2 = 4^2 + 3^2 + 1^2$.

(1 mark for to show Factors & 3 marks expressions)

Answer

100, 110, 101, 302, 320, 230, 203, 431, 413, 314, 341, 134, 143, 510, 501, 150, 105, ……..

(1 marks))

(b) a, b, c are distinct real numbers and there are real numbers x and y such that $a^3 + ax + y = 0$, $b^3 + bx + y = 0$ and $c^3 + cx + y = 0$.

Show that $a + b + c = 0$.

Soln.

Subtracting the first two equations and dividing by $(a - b)$ we get $(a^2 + ab + b^2) + x = 0$. (2)

Similarly we get $(b^2 + bc + c^2) + x = 0$. (2)

Subtracting we get $(a - c)(a + b + c) = 0$. Hence $a + b + c = 0$. (1)

4. Solve for 'x' and 'y' : $x+xy+y=11$, $x^2y+xy^2=30$

Soln. The original system of equations can be factored into

$(x+y)+xy=11$, $(x+y)xy=30$ (1)

Let $(x+y)$ and xy are roots of a quadratic equation whose sum of roots is '11' and product is '30'so the such equation may be let $z^2-11z+30=0$, which gives the roots as $z_1=6$ and $z_2=5$

(2+2)

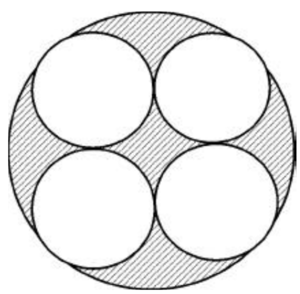
So $x+y=6$ and $xy=5$ or $x+y=5$ and $xy=6$, (1)

Which express two quadratic equations as under (2)

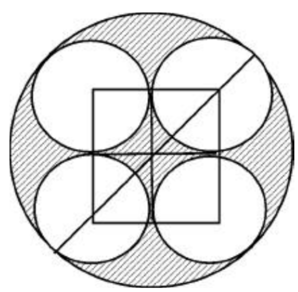
m2-6m+5=0, $m^2-5m+6=0$

The solution of system of equations are(2,3),(3,2) or (5,1),(1,5) (2)

5. Four small circles of radius 1 are tangent to each other and to a large circle containing them, as shown in the figure. What is the area of theregion inside the larger circle, but outside all the smaller circles ?



Solution:



From figure the radius of bigger circle is $\dfrac{2+2\sqrt{2}}{2}=1+\sqrt{2}$

(Construction-2 mark,To find Radius -2 marks)

So, area of bigger circle = $\pi\left(1+\sqrt{2}\right)^2$ (2)

Area inside four smaller circle = 4π (2)

So area of the region inside the larger circle, but outside all smaller circle

$= \pi(1+\sqrt{2})^2 - 4\pi$

$= \pi(2\sqrt{2}-1)$ (2)

6. (a) Find a natural number 'n' such that $3^? + 3^{??} + 3^{??} + 3^?$ is a perfect cube of an integer.

Soln.Given $3^9 + 3^{12} + 3^{15} + 3^n = 3^9[1+3^3+3^6+3^{n-9}]$ (1)

⇨ $(3^3)^3[1+3.3^2+3(3^2)^2+(3^2)^3+3^{n-9}-3(3^2)^2]$ by adding and subtracting $3(3^2)^2$ (2)

⇨ $(3^3)^3[(1+3^2)^3]$ provided $3^{n-9} - 3^5 = 0$ (1)

⇨ n=14 (1)

6 (b) The sum of two positive integers is 52 and their LCM is 168. Find the numbers.

Soln. Let the GCD of the numbers be d so the numbers are of the form dp and dq where p and q are some integers prime to each other.

Given that the sum of numbers is 52 hence

dp+dq=52 (1)

Also LCM of numbers is 168 hence dpq=168 (2)

From (1) and (2) 42(p+q)=13pq

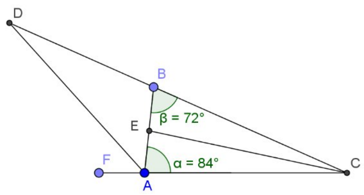
Since p and q are prime to each other, therefore p and q both are prime to p+q i.e pq is prime to p+q

Therefore pq divides 42 also 42 being prime to 13 , it follows that 42 divides pq .

Since the positive integers 42 and pq divides each other , therefore pq=42 and p+q-13, Therefore p and q are roots of $x^2 - 13x + 42 = 0$ so that p and q are 6,and 7 So d=168/pq= 4 so the numbers are 24 and 28. (2)

7.In triangle ABC, ∠ ABC=72^0 , ∠ CAB=$84^0$and the point E lies on AB so that EC bisects angle BCA. The point F lies on CA extended and the point D lies on CB extended so that DA bisects ∠ BAF.
Prove that AD=CE .

Soln.



(Construction-1 mark)

EC bisects angle ∠BCA so ∠ BCE=∠ECA=>∠BCA= 180^0-72^0-84^0= 24^0

So ∠ BCE=24^0/2=12^0

∠BEC= 180^0-72^0-12^0=96^0

So ∠CEA= 180^0-12^0-84^0 =84^0

i.e. ∠CEA=∠CAE-84^0 so Triangle CEA is an isosceles . (To find angle & showing isosceles -4 marks)

=> CE=CA (1)

∠EAF=180^0-84^0= 96^0

Given that AD bisects ∠EAF so ∠EAD=∠DAF= 96/2= 48^0

Now ∠CAD= 84^0+48^0= 132^0

∠ BDA- 180^0-132^0-24^0=24^0

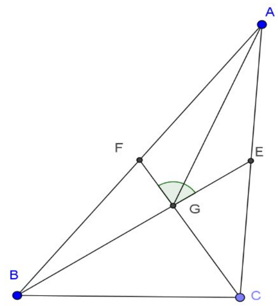
So triangle CDA is an isosceles => AD=CA (2) (To find angle & showing isosceles -4 marks)

From (1) and(2) we conclude that AD=CE

Proved (1)

8. (a) In triangle ABC, BC and CF are medians , BE= 9 cm. and CF=12 cm.
If BE is perpendicular to CF, find the area of triangle ABC.

Soln.



(Construction-1 mark)

'G' is the Centroid of the triangle ABC ,so BG/GE=AG/GE = 2/1

so BG=6 (1)

& GE=3 units similarly CG=8 and GF=4 (1)

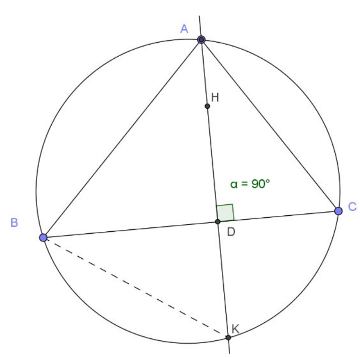
Area of triangle ABC= 3 x area of triangle BGC .

Now area of triangle BGC= 1/2xBGxGC= 1/2x 6x8= 24 sq. units (1)

Now area triangle= 3x24=72 sq units (1)

(b) The altitude AD of triangle ABC is produced to cut its circumcircle in 'K'. Prove that HD=DK, where 'H' is the orthocenter.

Soln.



(construction-1 mark)

∠KBC=∠KEC (Angle in same segment) (1)

But ∠KAC = 90^0-∠C (1)

∠KBC=∠ EBC (1)

Therefore Triangle HBD ≅ ???????? ? ? ? — (1)

Therefore HD=DK

9. (a) Three fair dice are rolled together at the same time . Find the probability that the product of the three numbers appearing on their tops is a prime number.

Soln.

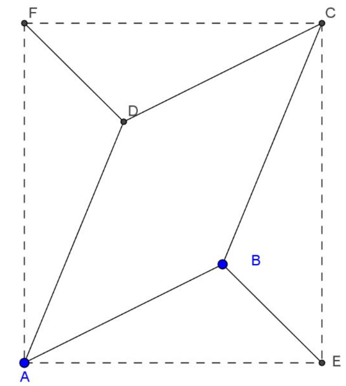
Total number of outcomes =6X6X6=216 (1)

Favorable outcomes (1,2,1),(1,3,1),(1,5,1),(2,1,1) ,(3,1,1),(5,1,1),(1,12) ,(1,1,3),(1,1,5) (2)

Total no. of outcomes=9 (1)

Required probability= $\frac{?}{???}$= $\frac{?}{??}$ (1)

(b) The area of a parallelogram is 600 sq units.The coordinates of A,B and D are (0,0),(20,10) and (10,y) respectively. Find the value of 'y'.



Soln.

Coordinates of 'C' =(adding x -coordinates of 'B and 'D', and adding y coordinates of B & D (30,y+10)

(1)

Area of parallelogram = Area of rectangle AECF- (sum of areas of triangleAEB,triangle EBC, triangleCDF and triangle ADF) (1)

= 30(y+10)-1/2(30x10+10(y+10)+30x10+10(y+10) (2)

=20y-100=600 (given)

= y=35 (1)

10. (a) If a+b$\sqrt{3}$ = $\sqrt{\frac{6+2\sqrt{3}}{33-19\sqrt{3}}}$, find 'a'and'b'

Soln.

$\sqrt{\frac{(2\sqrt{3}+2)}{(11\sqrt{3}-19)}}$

= $\sqrt{\frac{(2\sqrt{3}+2)}{(11\sqrt{3}-19)}}$ (1)

= (2)

$$\sqrt{\frac{2(\sqrt{3}+1)(11\sqrt{3}+19)}{363-361}}$$

$$\sqrt{(52+30\sqrt{3})}$$

=($\sqrt{(\sqrt{25}+\sqrt{27})}^2$ (1)

$$=5+3\sqrt{3}$$

so a=5, b=3 (1)

(b) Solve for x,y and z :

$x^2+xy+xz=-12$

$y^2+yz+xy=30$

$z^2+xz+yz=18$

Soln.

The given equations can be written as

x(x+y+z)= -12 (1)

y(x+y+z) = 30 (2)

z(x+y+z)= 18 (3)

adding all we get $(x+y+z)^2=36$ **(1+2)**

- ⇨ **x+y+z=é**
- ⇨ **if x+y+z=6 gives x=-2,y=5,z=3 if x+y+z=-6 gives x=2,y=-5,z=-3** **(1+1)**